ExerciseThinking

卷二 数学物理方程习题解

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目 录

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第一章 典型方程与定解问题

1.1 判断方程类型

1.1.1

例：$x^2u_{xx} - y^2u_{yy} = 0$.

$$a = x^2, b = 0, c = -y^2$$
$$\Delta = b^2 - ac = x^2y^2$$

若 x, y 其中一个为 0,则

$$\Delta = 0$$

方程为抛物型.

若 x, y 均不为 0,则

$$\Delta > 0$$

方程为双曲型.

1.1.2

例：$u_{xx} + xyu_{yy} = 0$.

$$a = 1, b = 0, c = xy$$
$$\Delta = b^2 - ac = -xy$$

若 x, y 其中一个为 0,则

$$\Delta = 0$$

方程为抛物型.

若 x, y 均不为 0 且异号,则

$$\Delta > 0$$

方程为双曲型.

若 x, y 均不为 0 且同号,则

$$\Delta < 0$$

方程为椭圆型.

1.2 化下列方程为标准形

1.2.1

例：$u_{xx} + 4u_{xy} + 5u_{yy} + u_x + u_y = 0$.

$$a = 1, b = 2, c = 5$$
$$\Delta = b^2 - ac = -1$$

有

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5 = 0$$

则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \pm i$$
$$\mathrm{d}y = (2 \pm i)\,\mathrm{d}x$$
$$y = (2 \pm i)x + C$$

则

$$\varphi(x,y)=2x-y\pm ix=C$$

取 $\xi=x,\eta=2x-y$,有

$$J=\frac{D(\xi,\eta)}{D(x,y)}=\det\left(\xi_x,\xi_y;\eta_x,\eta_y\right)=-1\neq 0$$
$$u_x=u_\xi\frac{\partial\xi}{\partial x}+u_\eta\frac{\partial\eta}{\partial x}=u_\xi+2u_\eta$$
$$u_y=u_\xi\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial\eta}{\partial y}=-u_\eta$$
$$u_{xx}=u_{\xi\xi}+4u_{\eta\xi}+4u_{\eta\eta}$$
$$u_{yy}=u_{\eta\eta}$$
$$u_{xy}=-u_{\eta\xi}-2u_{\eta\eta}$$

代入得

$$u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}-4u_{\xi\eta}-8u_{\eta\eta}+5u_{\eta\eta}+u_\xi+2u_\eta-u_\eta=0$$

化简得

$$u_{\xi\xi}+u_{\eta\eta}=-u_\xi-u_\eta$$

1.2.2

例：$u_{xx}-4u_{xy}+u_{yy}=0$.

$$a=1,b=-2,c=1$$
$$\Delta=b^2-ac=3$$

有

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+4\frac{\mathrm{d}y}{\mathrm{d}x}+1=0$$

则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2\pm\sqrt{3}$$
$$\mathrm{d}y=\left(-2\pm\sqrt{3}\right)\mathrm{d}x$$
$$y=\left(-2\pm\sqrt{3}\right)x+C$$

则

$$\varphi(x,y)=y+\left(2\pm\sqrt{3}\right)x=C$$

取 $\xi=y+\left(2+\sqrt{3}\right)x,\eta=y+\left(2-\sqrt{3}\right)x$,有

$$J=\frac{D(\xi,\eta)}{D(x,y)}=\det\left(\xi_x,\xi_y;\eta_x,\eta_y\right)=2\sqrt{3}\neq 0$$
$$u_x=u_\xi\frac{\partial\xi}{\partial x}+u_\eta\frac{\partial\eta}{\partial x}=\left(2+\sqrt{3}\right)u_\xi+\left(2-\sqrt{3}\right)u_\eta$$
$$u_y=u_\xi\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial\eta}{\partial y}=u_\xi+u_\eta$$
$$u_{xx}=\left(7+4\sqrt{3}\right)u_{\xi\xi}+2u_{\eta\xi}+\left(7-4\sqrt{3}\right)u_{\eta\eta}$$
$$u_{yy}=u_{\xi\xi}+u_{\eta\eta}+2u_{\xi\eta}$$
$$u_{xy}=\left(2+\sqrt{3}\right)u_{\xi\xi}+\left(2-\sqrt{3}\right)u_{\eta\eta}+4u_{\eta\xi}$$

代入化简得

$$\left(7+4\sqrt{3}\right)u_{\xi\xi}+2u_{\eta\xi}+\left(7-4\sqrt{3}\right)u_{\eta\eta}+u_{\xi\xi}+u_{\eta\eta}+2u_{\xi\eta}-4\left[\left(2+\sqrt{3}\right)u_{\xi\xi}+\left(2-\sqrt{3}\right)u_{\eta\eta}+4u_{\eta\xi}\right]=0$$

即

$$u_{\xi\eta}=0$$

1.3 求四种不同边值条件下对应的固有值问题的解

1.3.1

例：$T''+\lambda a^2T=0,X''+\lambda X=0,X(0)=0,X(l)=0$.

当 $\lambda < 0$,

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

由 $X(0) = 0, X(l) = 0$得

$$c_1 + c_2 = 0, c_1 e^{\sqrt{-\lambda}l} + c_2 e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0$,

$$X(x) = c_1 x + c_2$$

由 $X(0) = 0, X(l) = 0$得

$$c_2 = 0, c_1 l + c_2 = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda > 0$,

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

由 $X(0) = 0, X(l) = 0$得

$$c_1 = 0, c_2 \sin\sqrt{\lambda}l = 0$$

避免平凡解后解得

$$c_1 = 0, c_2 = 1, \lambda_k = \left(\frac{k\pi}{l}\right)^2, k = 1, 2, \ldots$$

将 $\lambda_k = \left(\frac{k\pi}{l}\right)^2$ 代入 $X(x)$得

$$X_k(x) = \sin\frac{k\pi}{l}x$$

同样的有

$$T_k''(t) + \left(\frac{k\pi}{l}a\right)^2 T_k(t) = 0$$

由于 $\left(\frac{k\pi}{l}a\right)^2 > 0$,则

$$T_k(x) = A_k \cos\frac{k\pi}{l}at + B_k \sin\frac{k\pi}{l}at$$
$$u_k(x,t) = \left(A_k \cos\frac{k\pi}{l}at + B_k \sin\frac{k\pi}{l}at\right)\sin\frac{k\pi}{l}x$$

由线性叠加原理得

$$u(x,t) = \sum_{k=1}^{\infty}\left(A_k \cos\frac{k\pi}{l}at + B_k \sin\frac{k\pi}{l}at\right)\sin\frac{k\pi}{l}x$$

其中

$$A_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$
$$B_k = \frac{2}{k\pi a}\int_0^l \psi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

1.3.2

例: $T'' + \lambda a^2 T = 0, X'' + \lambda X = 0, X'(0) = 0, X'(l) = 0$.

当 $\lambda < 0$,

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$
$$X'(x) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

由 $X'(0) = 0, X'(l) = 0$得

$$c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = 0, c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} = 0$$

解得

$$c_1 = c_2 = 0$$

当 $\lambda = 0$,

$$X(x)=c_1x+c_2$$
$$X'(x)=c_1$$

由 $X'(0)=0,X'(l)=0$ 得

$$c_1=0$$

解得

$$X(x)=1$$

当 $\lambda>0$,

$$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$$
$$X'(x)=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x+c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

由 $X'(0)=0,X'(l)=0$ 得

$$c_2=0,-c_1\sqrt{\lambda}\sin\sqrt{\lambda}l=0$$

避免平凡解后解得

$$c_1=1,c_2=0,\lambda_k=\left(\frac{k\pi}{l}\right)^2,k=1,2,...$$

将 $\lambda_k=\left(\frac{k\pi}{l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x)=\cos\frac{k\pi}{l}x$$

将 $\lambda_k=\left(\frac{k\pi}{l}\right)^2,k=0,1,2,...$ 代入 $T''+\lambda a^2T=0$ 得

$$T_0(x)=A_0+B_0t,k=0$$
$$T_k(x)=A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at,k=1,2,...$$

由线性叠加原理得

$$u(x,t)=A_0+B_0t+\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at\right)\cos\frac{k\pi}{l}x$$

其中

$$A_0=\frac{1}{l}\int_0^l\varphi(x)\,\mathrm{d}x$$
$$B_0=\frac{1}{l}\int_0^l\psi(x)\,\mathrm{d}x$$
$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x$$
$$B_k=\frac{2}{k\pi a}\int_0^l\psi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x$$

可以改写为

$$u(x,t)=\frac{1}{2}(A_0+B_0t)+\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at\right)\cos\frac{k\pi}{l}x$$
$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x,k=0,1,2,...$$
$$B_k=\frac{2}{k\pi a}\int_0^l\psi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x,k=1,2,...$$
$$B_0=\frac{2}{l}\int_0^l\psi(x)\,\mathrm{d}x$$

1.3.3

例：$X''+\lambda X=0,X'(0)=0,X(l)=0.$

当 $\lambda<0$,

$$X(x)=c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}$$
$$X'(x)=c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x}-c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

由 $X'(0)=0,X(l)=0$ 得

$$c_1\sqrt{-\lambda}-c_2\sqrt{-\lambda}=0,c_1e^{\sqrt{-\lambda}l}+c_2e^{-\sqrt{-\lambda}l}=0$$

解得

$$c_1=c_2=0$$

当 $\lambda=0$,

$$X(x)=c_1x+c_2$$
$$X'(x)=c_1$$

由 $X'(0)=0, X(l)=0$ 得

$$c_1=0, c_1l+c_2=0$$

解得

$$c_1=c_2=0$$

当 $\lambda>0$,

$$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$$
$$X'(x)=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x+c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

由 $X'(0)=0, X(l)=0$ 得

$$c_2=0, c_1\cos\sqrt{\lambda}l=0$$

避免平凡解后解得

$$c_1=1, c_2=0, \lambda_k=\left(\frac{(2k-1)\pi}{2l}\right)^2, k=1,2,\dots$$

将 $\lambda_k=\left(\frac{(2k-1)\pi}{2l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x)=\cos\frac{(2k-1)\pi}{2l}x$$

将 $\lambda_k=\left(\frac{k\pi}{l}\right)^2, k=0,1,2,\dots$ 代入 $T''+\lambda a^2T=0$ 得

$$T_k(x)=A_k\cos\frac{(2k-1)\pi}{2l}at+B_k\sin\frac{(2k-1)\pi}{2l}at, k=1,2,\dots$$

由线性叠加原理得

$$u(x,t)=\sum_{k=1}^{\infty}\left(A_k\frac{(2k-1)\pi}{2l}at+B_k\sin\frac{(2k-1)\pi}{2l}at\right)\cos\frac{(2k-1)\pi}{2l}x$$

其中

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\frac{(2k-1)\pi}{2l}x\,\mathrm{d}x$$
$$B_k=\frac{4}{(2k-1)\pi a}\int_0^l\psi(x)\cos\frac{(2k-1)\pi}{2l}x\,\mathrm{d}x$$

1.3.4

例: $X''+\lambda X=0, X(0)=0, X'(l)=0.$

当 $\lambda<0$,

$$X(x)=c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}$$
$$X'(x)=c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x}-c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

由 $X(0)=0, X'(l)=0$ 得

$$c_1+c_2=0, c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l}-c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l}=0$$

解得

$$c_1=c_2=0$$

当 $\lambda=0$,

$$X(x)=c_1x+c_2$$
$$X'(x)=c_1$$

由 $X(0)=0, X'(l)=0$ 得

$$c_1=c_2=0$$

当 $\lambda>0$,

$$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$$
$$X'(x)=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x+c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

由 $X(0)=0, X'(l)=0$ 得

$$c_1=0, c_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0$$

避免平凡解后解得

$$c_1=0, c_2=1, \lambda_k=\left(\frac{(2k-1)\pi}{2l}\right)^2, k=1,2,...$$

将 $\lambda_k=\left(\frac{(2k-1)\pi}{2l}\right)^2$ 代入 $X(x)$ 得

$$X_k(x)=\sin\frac{(2k-1)\pi}{2l}x$$

将 $\lambda_k=\left(\frac{k\pi}{l}\right)^2, k=0,1,2,...$ 代入 $T''+\lambda a^2T=0$ 得

$$T_k(x)=A_k\cos\frac{(2k-1)\pi}{2l}at+B_k\sin\frac{(2k-1)\pi}{2l}at, k=1,2,...$$

由线性叠加原理得

$$u(x,t)=\sum_{k=1}^{\infty}\left(A_k\frac{(2k-1)\pi}{2l}at+B_k\sin\frac{(2k-1)\pi}{2l}at\right)\sin\frac{(2k-1)\pi}{2l}x$$

其中

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{(2k-1)\pi}{2l}x\,\mathrm{d}x$$
$$B_k=\frac{4}{(2k-1)\pi a}\int_0^l\psi(x)\sin\frac{(2k-1)\pi}{2l}x\,\mathrm{d}x$$

1.4 解下列定解问题

1.4.1

例: $\begin{cases}u_t=a^2u_{xx}\\ u|_{t=0}=\varphi(x)\\ u_x|_{x=0}=u_x|_{x=l}=0\end{cases}$

令 $u(x,t)=X(x)T(t)$

有

$$X''+\lambda X=0, X'(0)=X'(l)=0$$
$$T'+a^2\lambda T=0$$

$\lambda<0$,无非零解

$\lambda=0$,解得 $X_0=A_0$

$\lambda>0$,

$$X=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$$

由 $X'(0)=X'(l)=0$

$$\lambda_k=\left(\frac{k\pi}{l}\right)^2$$

则

$$X_k(x)=A_k\cos\frac{k\pi}{l}x$$

$\lambda=0$,解得 $T_0=B_0, u_0=X_0T_0=A_0B_0=C_0$

$\lambda>0$

$$T_{k'}+\frac{a^2k^2\pi^2}{l^2}T_k=0$$
$$T_k(t)=B_ke^{-\frac{a^2k^2\pi^2}{l^2}t}$$
$$u_k=X_kT_k=C_ke^{-\frac{a^2k^2\pi^2}{l^2}t}\cos\frac{k\pi}{l}x$$

由叠加原理

$$u(x,t)=C_0+\sum_{k=1}^{\infty}C_ke^{-\frac{a^2k^2\pi^2}{l^2}t}\cos\frac{k\pi}{l}x$$

由 $u(x,0)=\varphi(x)=C_0+\sum_{k=1}^{\infty}C_k\cos\frac{k\pi}{l}x$

$$C_0=\frac{1}{l}\int_0^l\varphi(x)\,\mathrm{d}x$$
$$C_k=\frac{2}{l}\int_0^l\varphi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x$$

1.4.2

例：$\begin{cases} u_{tt}=a^2u_{xx} \text{ if } 0<x<1 \mid t>0 \\ u|_{t=0}=\begin{cases} x \text{ if } 0\le x\le \frac{1}{2} \\ -x \text{ if } \frac{l}{2}\le x\le 1 \end{cases} \\ u_t|_{t=0}=x(1-x) \text{ if } 0\le x\le 1 \\ u|_{x=0}=u|_{x=1}=0 \text{ if } t\ge 0 \end{cases}$

令 $u(x,t)=X(x)T(t)$

$$X''+\lambda X=0$$
$$T''+a^2\lambda T=0$$
$$X(0)T(t)=X(1)T(t)=0$$

$\lambda\le 0$,只有平凡零解

$\lambda>0$

$$\lambda_k=(k\pi)^2$$
$$X_k(x)=\sin k\pi x$$
$$T_k(t)=A_k\cos k\pi at+B_k\sin k\pi at$$
$$u(x,t)=\sum_{k=1}^{\infty}(A_k\cos k\pi at+B_k\sin k\pi at)\sin\frac{k\pi}{l}x$$

其中

$$A_k=2\left(\int_0^{\frac{1}{2}}x\sin k\pi x\,\mathrm{d}x-\int_{\frac{1}{2}}^1 x\sin k\pi x\,\mathrm{d}x\right)$$
$$=\frac{2}{k\pi}\left(-\int_0^{\frac{1}{2}}x\,\mathrm{d}\cos k\pi x+\int_{\frac{1}{2}}^1 x\,\mathrm{d}\cos k\pi x\right)$$
$$=\frac{2}{k\pi}\left[\int_0^{\frac{1}{2}}\cos k\pi x\,\mathrm{d}x-\int_{\frac{1}{2}}^1\cos k\pi x\,\mathrm{d}x+(-1)^k\right]$$
$$=\frac{2}{k\pi}\left[\frac{4}{k\pi}(-1)^{k+1}+(-1)^k\right]$$
$$=\frac{2}{k\pi}(-1)^k\left(1-\frac{4}{k\pi}\right)$$
$$B_k=\frac{2}{k\pi a}\int_0^1 x(1-x)\sin k\pi x\,\mathrm{d}x$$
$$=-\frac{2}{k^2\pi^2a}\int_0^1 x(1-x)\,\mathrm{d}\cos k\pi x$$
$$=\frac{2}{k^2\pi^2a}\int_0^1(1-2x)\cos k\pi x\,\mathrm{d}x$$
$$=-\frac{4}{k^3\pi^3a}\int_0^1 x\,\mathrm{d}\sin k\pi x$$
$$=\frac{4}{k^4\pi^4a}[1-(-1)^k]$$

1.5 相容性条件

1.5.1

例：$\begin{cases} u_{tt}=a^2u_{xx} \text{ if } 0<x<l \mid t>0 \\ u|_{t=0}=\varphi(x).u_t|_{t=0}=\psi(x) \text{ if } 0\le x\le l \\ u_x|_{x=0}=u_x|_{x=1}=0 \text{ if } t\ge 0 \end{cases}$

答：进行分离变量后得到固有值问题

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(l)=0 \end{cases}$$

由例题 1.1.2 可知

$$u_0(x,t)=\frac{1}{2}(A_0+B_0t)$$
$$u_k(x,t)=\left(A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at\right)\cos\frac{k\pi}{l}x,k=1,2,...$$
$$u(x,t)=\frac{1}{2}(A_0+B_0t)+\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at\right)\cos\frac{k\pi}{l}x$$

其中

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x, k=0,1,2,...$$
$$B_k=\frac{2}{k\pi a}\int_0^l\psi(x)\cos\frac{k\pi}{l}x\,\mathrm{d}x, k=1,2,...$$
$$B_0=\frac{2}{l}\int_0^l\psi(x)\,\mathrm{d}x$$

则解的存在性条件为：

$$\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi}{l}at+B_k\sin\frac{k\pi}{l}at\right)\cos\frac{k\pi}{l}x$$
$$\sum_{k=1}^{\infty}(u_k)_t, \sum_{k=1}^{\infty}(u_k)_{tt}, \sum_{k=1}^{\infty}(u_k)_x, \sum_{k=1}^{\infty}(u_k)_{xx}$$

一致收敛，其优级数为

$$\sum_{k=1}^{\infty}k^m(|A_k|+|B_k|), m=0,1,2$$

若 $\varphi(x)\in C^2, \psi(x)\in C^1, \varphi^{(3)}(x)$与 $\psi''(x)$分段连续,且 $\varphi(0)=\psi(0)=0, \varphi''(0)=\varphi''(l)=0, \psi(0)=\psi(l)=0$,根据引理 3.1,上述优级数收敛,则例题 3.1 的解可以表示为形如 $u(x,t)$的级数.

第二章 分离变量法

2.1 解如下定解问题

2.1.1

例：$u_{xx}+u_{yy}=0, 0<x<a, 0<y<b$

$u|_{x=0}=u|_{x=a}=0, 0\le y\le b$

$u_y|_{y=0}=f(x), u|_{y=b}=0, 0\le x\le a$

$$u(x,y)=X(x)Y(y)$$

将其代入分离变量可得固有值问题

$$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X(a)=0\end{cases}$$

当 $\lambda\le 0$,无非零解

当 $\lambda>0$,有

$$\lambda_k=\left(\frac{k\pi}{a}\right)^2, X(x)=\sin\frac{k\pi}{a}x, k=1,2,\ldots$$

将 λ_k 代入关于 Y 的方程

$$Y''(y)-\left(\frac{k\pi}{a}\right)^2Y(y)=0$$

解得

$$Y(y)=A_k ch\frac{k\pi}{a}y+B_k sh\frac{k\pi}{a}y$$

则

$$u(x,y)=\sum_1^\infty\left(A_k ch\frac{k\pi}{a}y+B_k sh\frac{k\pi}{a}y\right)\sin\frac{k\pi}{a}x$$

由边界条件得

$$f(x)=\sum_1^\infty B_k\frac{k\pi}{a}\sin\frac{k\pi}{a}x$$
$$\sum_1^\infty\left(A_k ch\frac{k\pi b}{a}+B_k sh\frac{k\pi b}{a}\right)\sin\frac{k\pi}{a}x=0$$

其中

$$A_k=\frac{2}{k\pi}\int_0^a f(x)\sin\frac{k\pi}{a}x\,\mathrm{d}x, k=1,2,\ldots$$

相应的

$$B_k=-\frac{2ch\frac{k\pi b}{a}}{k\pi sh\frac{k\pi b}{a}}\int_0^l\varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

2.1.2

例：$u_{xx}+u_{yy}=f(x), 0<x<a, 0<y<b$

$u|_{x=0}=A, u|_{x=a}=B, 0\le y\le b$

$u_y|_{y=0}=g(x), u|_{y=b}=0, 0\le x\le a$

$$u(x,y)=v(x,y)+w(x,y)$$

取

$$w(x,y)=\frac{B-A}{a}x+A$$

则有

$$v_{xx}+v_{yy}=f(x,y)$$
$$v|_{x=0}=v_t|_{x=a}=0$$
$$v_y|_{y=0}=g_1(x), v|_{y=b}=g_2(x)$$

其中

$$f(x,y)=f(x)+\frac{A-B}{a}x-A$$

$$g_1(x) = g(x)$$
$$g_2(x) = \frac{A-B}{a}x - A$$

设 $v(x,y) = \sum_1^\infty v_k(y)\sin\frac{k\pi}{a}x$ 得

$$v_k''(y) - \left(\frac{k\pi}{a}\right)^2 v_{k(y)} = f_k(y)$$
$$v_k'(0) = g_{1k}(x), v_k(b) = g_{2k}(x)$$

解得

$$v_k(y) = A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau$$

则

$$v(x,y) = \sum_1^\infty \left[A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau\right]\sin\frac{k\pi}{a}x$$

整理得

$$u(x,y) = \sum_1^\infty \left[A_k ch\frac{k\pi}{a}y + B_k sh\frac{k\pi}{a}y + \frac{a}{k\pi}\int_0^y f_{k(\tau)} sh\frac{k\pi}{a}(y-\tau)\,\mathrm{d}\tau\right]\sin\frac{k\pi}{a}x$$
$$+\frac{B-A}{a}x + A$$

其中 A_k, B_k 或许老天知道

2.2 解下列定解方程

2.2.1

例: $\begin{cases} u_t = a^2 u_{xx} \text{ if} 0<x<l.t>0 \\ u|_{t=0} = 0 \text{ if} 0\le x\le l \\ u_x|_{x=0} = \mu_1(x).u_x|_{x=l} = \mu_2(x) \text{ if } t\ge 0 \end{cases}$

答: 令

$$u(x,t) = v(x,t) + w(x,t)$$

将其代入边值条件

$$\begin{cases} \mu_1(t) = u_x|_{x=0} = v_x|_{x=0} + w_x|_{x=0} \\ \mu_2(t) = u_x|_{x=l} = v_x|_{x=l} + w_x|_{x=l} \end{cases}$$

为使 $v_x|_{x=0} = v_x|_{x=l} = 0$,必须有

$$w_x|_{x=0} = \mu_1(t), w_x|_{x=l} = \mu_2(t)$$

取

$$w_x(x,t) = \frac{l-x}{l}\mu_1(t) + \frac{x}{l}\mu_2(t)$$
$$w(x,t) = -\frac{(l-x)^2}{2l}\mu_1(t) + \frac{x^2}{2l}\mu_2(t)$$

此时 $v(x,t)$满足以下定解问题

$$\begin{cases} v_t = a^2 v_{xx} + f(x,t) \text{ if} 0<x<l.t>0 \\ v|_{t=0} = \varphi(x) \text{ if} 0\le x\le l \\ u_x|_{x=0} = u_x|_{x=l} = 0 \text{ if } t\ge 0 \end{cases}$$

其中

$$f(x,t) = -\frac{a^2}{l}\mu_1(t) + \frac{a^2}{l}\mu_2(t) + \frac{(l-x)^2}{2l}\mu_1'(t) - \frac{x^2}{2l}\mu_2'(t)$$
$$\varphi(x) = \frac{(l-x)^2}{2l}\mu_1(0) - \frac{x^2}{2l}\mu_2(0)$$

不妨令

$$v(x,t) = \sum_{k=1}^\infty v_k(t)\sin\frac{k\pi}{l}x$$

代入定解方程有

$$\sum_{k=1}^\infty v_k'(t)\sin\frac{k\pi}{l}x = -a^2\sum_{k=1}^\infty \left(\frac{k\pi}{l}\right)^2 v_k(t)\sin\frac{k\pi}{l}x$$
$$+\sum_{k=1}^\infty f_k(t)\sin\frac{k\pi}{l}x$$
$$\sum_{k=1}^\infty v_k(0)\sin\frac{k\pi}{l}x = +\sum_{k=1}^\infty \varphi_k\sin\frac{k\pi}{l}x$$

其中

$$f_k(t)=\frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$
$$\varphi_k=\frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

解得

$$v_k(t)=\varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t}+\int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau$$

则

$$v(x,t)=\sum_{k=1}^{\infty}\left[\varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t}+\int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau\right]\sin\frac{k\pi}{l}x$$

相应的

$$u(x,t)=\sum_{k=1}^{\infty}\left[\varphi_k e^{-\left(\frac{k\pi a}{l}\right)^2 t}+\int_0^t f_k(\tau)e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau\right]\sin\frac{k\pi}{l}x$$
$$-\frac{(l-x)^2}{2l}\mu_1(t)+\frac{x^2}{2l}\mu_2(t)$$

2.2.2

例: $\begin{cases} u_{tt}=a^2u_{xx}+bu \text{ if} 0<x<l.t>0 \\ u|_{t=0}=u_t|_{t=0}=0 \text{ if} 0\le x\le l \\ u|_{x=0}=\mu_1(x).u|_{x=l}=\mu_2(x) \text{ if } t\ge 0 \end{cases}$.

答: 令

$$u(x,t)=v(x,t)+w(x,t)$$

取

$$w(x,t)=\frac{l-x}{l}\mu_1(t)+\frac{x}{l}\mu_2(t)$$

则有

$$v_{tt}=a^2v_{xx}+bv+f(x,t)$$
$$v|_{t=0}=\varphi(x),v_t|_{t=0}=\psi(x)$$
$$v|_{x=0}=v|_{x=l}=0$$

其中

$$f(x,t)=\frac{l-x}{l}\mu_1(t)+\frac{x}{l}\mu_2(t)-\frac{l-x}{l}\mu_1''(t)-\frac{x}{l}\mu_2''(t)$$
$$\varphi(x)=-\frac{l-x}{l}\mu_1(0)-\frac{x}{l}\mu_2(0)$$
$$\psi(x)=-\frac{l-x}{l}\mu_1'(0)-\frac{x}{l}\mu_2'(0)$$

先考虑排除 $f(x,t)$ 的齐次方程,令 $v_1(x,t)=X(x)T(t)$ 分离变量得

$$\frac{T''}{a^2T}=\frac{X''}{x}+\frac{b}{a^2}=-\lambda$$

解得

$$\lambda_k+\frac{b}{a^2}=\left(\frac{k\pi}{l}\right)^2,k=1,2,\dots$$
$$X_k(x)=\sin\frac{k\pi}{l}x$$
$$T_k(t)=A_k\cos\omega_k x+B_k\sin\omega_k x$$

其中

$$\omega_k=\sqrt{\left(\frac{k\pi a}{l}\right)^2-b}$$
$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\omega_k x\,\mathrm{d}x$$
$$B_k=\frac{2}{\omega_k l}\int_0^l\psi(x)\sin\omega_k x\,\mathrm{d}x$$
$$v_1(x,t)=\sum_{k=1}^{\infty}(A_k\cos\omega_k x+B_k\sin\omega_k x)\sin\frac{k\pi}{l}x$$

由齐次化原理处理 $f(x,t)$ 得

$$v_2(x,t)=\sum_{k=1}^{\infty}\frac{2}{\omega_k l}\int_0^t\int_0^l f(\xi,\tau)\sin\frac{k\pi}{l}\xi\sin\omega_k(t-\tau)\,\mathrm{d}\xi\,\mathrm{d}x\sin\frac{k\pi}{l}x$$
$$v(x,t)=v_1(x,t)+v(x,t)$$
$$=\sum_{k=1}^{\infty}\left(A_k\cos\omega_k x+B_k\sin\omega_k x+\frac{2}{\omega_k l}\int_0^t\int_0^l f(\xi,\tau)\sin\frac{k\pi}{l}\xi\sin\omega_k(t-\tau)\,\mathrm{d}\xi\,\mathrm{d}x\right)\sin\frac{k\pi}{l}x$$

最后

$$u(x,t) = v(x,t) + w(x,t)$$

$$= \sum_{k=1}^{\infty} \left(A_k \cos \omega_k x + B_k \sin \omega_k x + \frac{2}{\omega_k l} \int_0^t \int_0^l f(\xi,\tau) \sin \frac{k\pi}{l} \xi \sin \omega_k (t-\tau) \,\mathrm{d}\xi \,\mathrm{d}x \right) \sin \frac{k\pi}{l} x$$

$$+ \frac{l-x}{l} \mu_1(t) + \frac{x}{l} \mu_2(t)$$

第三章 积分变换法

3.1 求下列函数的 Fourier 变换

3.1.1

例：$f(x)=\begin{cases}|x| \text{ if } x\leq a\\ 0 \text{ if } t>0\end{cases}$.

答:

$$\begin{aligned}\mathcal{F}[f(x)] &= \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x\\ &= \int_{-a}^{a}|x|e^{-i\lambda x}\,\mathrm{d}x\\ &= \int_0^a xe^{-i\lambda x}\,\mathrm{d}x+\int_{-a}^{0}-xe^{-i\lambda x}\,\mathrm{d}x\\ &= \int_0^a x\left(e^{-i\lambda x}+e^{i\lambda x}\right)\mathrm{d}x\\ &= 2\int_0^a x\cos\lambda x\,\mathrm{d}x\\ &= \frac{2a\sin\lambda a}{\lambda}+\frac{2(\cos\lambda a-1)}{\lambda^2}\end{aligned}$$

3.1.2

例：$f(x)=\cos\eta x^2$.

答:

$$\begin{aligned}\mathcal{F}[f(x)] &= \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x\\ &= \int_{-\infty}^{+\infty}\cos\eta x^2 e^{-i\lambda x}\,\mathrm{d}x\\ &= \frac{1}{2}\left(\int_{-\infty}^{+\infty}e^{-i\lambda x-i\eta x^2}\,\mathrm{d}x+\int_{-\infty}^{+\infty}e^{-i\lambda x+i\eta x^2}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(\int_{-\infty}^{+\infty}e^{-i\eta\left(x+\frac{\lambda}{2\eta}\right)^2+i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x+\int_{-\infty}^{+\infty}e^{i\eta\left(x-\frac{\lambda}{2\eta}\right)^2-i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(e^{i\frac{\lambda^2}{4\eta}}\int_{-\infty}^{+\infty}e^{-i\eta x^2}\,\mathrm{d}x+e^{-i\frac{\lambda^2}{4\eta}}\int_{-\infty}^{+\infty}e^{i\eta x^2}\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(e^{i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{i\eta}}+e^{-i\frac{\lambda^2}{4\eta}}\sqrt{-\frac{\pi}{i\eta}}\right)\end{aligned}$$

其中

$$\sqrt{\frac{\pi}{i\eta}}=\sqrt{\frac{\pi}{e^{-i\frac{\pi}{2}}\eta}}=\sqrt{\frac{\pi}{\eta}}e^{i\frac{\pi}{4}}$$
$$\sqrt{-\frac{\pi}{i\eta}}=\sqrt{\frac{\pi}{e^{i\frac{\pi}{2}}\eta}}=\sqrt{\frac{\pi}{\eta}}e^{-i\frac{\pi}{4}}$$
$$\mathcal{F}[f(x)]=\frac{1}{2}\sqrt{\frac{\pi}{\eta}}\left(e^{i\frac{\lambda^2}{4\eta}+i\frac{\pi}{4}}+e^{-i\frac{\lambda^2}{4\eta}-i\frac{\pi}{4}}\right)$$
$$\mathcal{F}[f(x)]=\frac{\sqrt{2}}{2}\sqrt{\frac{\pi}{i\eta}}(1+i)\sin\left(\frac{\lambda^2}{4\eta}+\frac{\pi}{4}\right)$$

3.1.3

例：$f(x)=\sin\eta x^2$.

答:

$$\mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$
$$= \int_{-\infty}^{+\infty} \sin\eta x^2 e^{-i\lambda x}\,\mathrm{d}x$$
$$= \frac{1}{2i}\left(\int_{-\infty}^{+\infty} e^{-i\lambda x + i\eta x^2}\,\mathrm{d}x - \int_{-\infty}^{+\infty} e^{-i\lambda x - i\eta x^2}\,\mathrm{d}x\right)$$
$$= \frac{\sqrt{2}}{2}\sqrt{\frac{\pi}{i\eta}}(i-1)\sin\left(\frac{\lambda^2}{4\eta} - \frac{\pi}{4}\right)$$

3.2 利用 Fourier 变换的性质求下列函数的 Fourier 变换

3.2.1

例: $f(x) = xe^{-a|x|}$.

答: 令 $g(x) = e^{-a|x|}$,易知 $\mathcal{F}[g(x)] = \frac{2a}{a^2+\lambda^2}$

$$\mathcal{F}[f(x)] = \mathcal{F}[xg(x)] = i\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[g(x)]$$
$$\mathcal{F}[f(x)] = -\frac{4a\lambda}{(a^2+\lambda^2)^2}$$

3.2.2

例: $f(x) = e^{-ax^2+ibx+c}$.

答: 令 $g(x) = e^{-ax^2}$,易知 $\mathcal{F}[g(x)] = \sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$

$$\mathcal{F}[f(x)] = e^c\mathcal{F}\left[e^{ibx}g(x)\right] = \sqrt{\frac{\pi}{a}}e^{-\frac{(\lambda-b)^2}{4a}+c}$$

3.3 求下列函数的 Fourier 逆变换

3.3.1

例: $F(\lambda) = e^{(-a^2\lambda^2+ib\lambda+c)t}$.

答:

$$f(x) = \mathcal{F}^{-1}[F(\lambda)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(-a^2\lambda^2+ib\lambda+c)t}e^{i\lambda x}\,\mathrm{d}\lambda$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2t\lambda^2+i(bt+x)\lambda+ct}\,\mathrm{d}\lambda$$

令 $u = \lambda - \frac{i(bt+x)}{2a^2t}$,则有

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2tu^2+\frac{(bt+x)^2}{4a^2t}+ct}\,\mathrm{d}u$$
$$= \frac{1}{2\pi}e^{\frac{(bt+x)^2}{4a^2t}+ct}\int_{-\infty}^{+\infty} e^{-a^2tu^2}\,\mathrm{d}u$$
$$= \frac{1}{2\pi}e^{\frac{(bt+x)^2}{4a^2t}+ct}\sqrt{\frac{\pi}{a^2t}}$$
$$f(x) = \frac{1}{\sqrt{4\pi a^2t}}e^{\frac{(bt+x)^2}{4a^2t}+ct}$$

3.4 用 Fourier 变换求解定解问题

3.4.1

$$u_t = a^2u_{xx} + bu_x + cu + f(x,t), -\infty < x < \infty, t > 0$$

例: $u|_{t=0} = 0$.

答: 对变量 x 进行 Fourier 变换, $U(\lambda,t) = \mathcal{F}[u(x,t)], F(\lambda,t) = \mathcal{F}[f(x,t)]$

$$U_t = (-a^2\lambda^2 + ib\lambda + c)U + F$$

解得

$$U = \left(-a^2\lambda^2 + ib\lambda + c\right) * F$$

$$u(x,t) = \mathcal{F}^{-1}\left[e^{(-a^2\lambda^2+ib\lambda+c)t} * F\right]$$

$$= \sqrt{\frac{\pi}{a^2}t}e^{\frac{(bt+x)^2}{4a^2t}+ct}f(x,t)$$

3.5 求下列函数的 Laplace 变换

3.5.1

例：$f(t) = t\cos\omega t$.

答:

$$\mathcal{L}[f(t)] = \int_0^\infty t\cos\omega t e^{-pt}\,\mathrm{d}t$$

$$= \frac{1}{p}\int_0^\infty e^{-pt}(\cos\omega t - \omega t\sin\omega t)\,\mathrm{d}t$$

$$= \frac{1}{p^2+\omega^2} - \frac{\omega}{p}\int_0^\infty t\sin\omega t e^{-pt}\,\mathrm{d}t$$

$$= \frac{1}{p^2+\omega^2} - \frac{\omega}{p^2}\int_0^\infty e^{-pt}(\sin\omega t + \omega t\cos\omega t)\,\mathrm{d}t$$

$$= \frac{1}{p^2+\omega^2} - \frac{\omega^2}{p^2}\frac{1}{p^2+\omega^2} - \frac{\omega^2}{p^2}\mathcal{L}[f(t)]$$

$$\mathcal{L}[f(t)] = \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$$

例：$f(t) = e^{\omega t}\cos\omega t$.

答:

$$\mathcal{L}[e^{\omega t}\cos\omega t] = \frac{p-\omega}{(p-\omega)^2+\omega^2}$$

3.6 求下列函数的 Laplace 逆变换

3.6.1

例：$F(p) = \frac{1}{(p^2+4)^2}$.

答:

$$\mathcal{L}^{-1}[F(p)] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{2}{p^2+2^2}\right] * \mathcal{L}^{-1}\left[\frac{2}{p^2+2^2}\right]$$

$$= \frac{1}{4}\int_0^t \sin 2t\sin 2(t-\tau)\,\mathrm{d}\tau$$

$$= \frac{1}{4}\sin 2t\int_0^t \sin 2u\,\mathrm{d}u$$

$$= -\frac{1}{8}\sin 2t(\cos 2t - 1)$$

$$= \frac{1}{8}\sin 2t - \frac{1}{16}\sin 4t$$

3.6.2

例：$F(p) = \frac{1}{p^4+5p^2+4}$.

答:

$$\mathcal{L}^{-1}[F(p)] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{p^2+1}\right] - \frac{1}{6}\mathcal{L}^{-1}\left[\frac{2}{p^2+2^2}\right]$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

3.7 用 Laplace 变换求下列常微分方程

3.7.1

例：$y'' + 2ky' + y = e^t, y(0) = 0, y'(0) = 1$.

答:

作 Laplace 变换有：

$$Y=\mathcal{L}[y]$$
$$\mathcal{L}[y']=p\mathcal{L}[y]-y(0)=pY$$
$$\mathcal{L}[y'']=p\mathcal{L}[y']-y'(0)=p^2Y-1$$

则有：

$$p^2Y-1+2kpY+Y=\tfrac{1}{p-1}$$
$$Y=\tfrac{p}{(p-1)(p^2+2kp+1)}$$

设

$$Y=\tfrac{A}{p-1}+\tfrac{Bp+c}{p^2+2kp+1}$$

解得

$$Y=\tfrac{1}{2k+2}\left(\tfrac{1}{p-1}+\tfrac{1}{p^2+2kp+1}-\tfrac{p}{p^2+2kp+1}\right)$$

1° $k=-1$

$$Y=\tfrac{1}{2k+2}\tfrac{p}{(p-1)^3}=\tfrac{1}{2k+2}\left(\tfrac{1}{(p-1)^2}+\tfrac{1}{(p-1)^3}\right)$$
$$y=\mathcal{L}^{-1}[Y]=\tfrac{1}{2k+2}\mathcal{L}^{-1}\left[\tfrac{1}{(p-1)^2}+\tfrac{1}{(p-1)^3}\right]$$
$$=\tfrac{1}{2k+2}e^t\left(t^2+\tfrac{1}{2}t^3\right)$$

2° $-1<k<1$

$$Y=\tfrac{1}{2k+2}\left[\tfrac{1}{p-1}+\tfrac{1+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}-\tfrac{p+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}\right]$$
$$y=\mathcal{L}^{-1}[Y]=\tfrac{1}{2k+2}\mathcal{L}^{-1}\left[\tfrac{1}{p-1}+\tfrac{1+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}-\tfrac{p+k}{(p+k)^2+\left(\sqrt{1-k^2}\right)^2}\right]$$
$$=\tfrac{1}{2k+2}\left[e^t+e^{-kt}\left(\tfrac{1+k}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}t-\cos\sqrt{1-k^2}t\right)\right]$$

2° $|k|\geq 1\setminus k=-1$

$$y=\mathcal{L}^{-1}[Y]=\tfrac{1}{2k+2}\mathcal{L}^{-1}\left[\tfrac{1}{p-1}+\tfrac{1+k}{(p+k)^2+\left(\sqrt{k^2-1}\right)^2}-\tfrac{p+k}{(p+k)^2+\left(\sqrt{k^2-1}\right)^2}\right]$$
$$=\tfrac{1}{2k+2}\mathcal{L}^{-1}\left[\tfrac{1}{p-1}+\tfrac{1+k}{2\sqrt{k^2-1}}\left(\tfrac{1}{p+k-\sqrt{k^2-1}}-\tfrac{1}{p+k+\sqrt{k^2-1}}\right)\right.$$
$$\left.-\tfrac{1}{2}\left(\tfrac{1}{p+k-\sqrt{k^2-1}}+\tfrac{1}{p+k+\sqrt{k^2-1}}\right)\right]$$
$$=\tfrac{1}{2k+2}\left\{e^t+\tfrac{k+1}{2\sqrt{k^2-1}}\left[e^{\left(\sqrt{k^2-1}-k\right)t}-e^{\left(-\sqrt{k^2-1}-k\right)t}\right]-\tfrac{1}{2}\left(e^{\left(\sqrt{k^2-1}-k\right)t}+e^{\left(-\sqrt{k^2-1}-k\right)t}\right)\right\}$$

3.7.2

例: $y''+4yt=k\cos\omega t, y(0)=0, y'(0)=0$.

答: 对 t 作 Laplace 变换有

$$Y=\mathcal{L}[y]$$
$$\mathcal{L}[y']=pY$$
$$\mathcal{L}[y']=p^2Y$$
$$\mathcal{L}[\cos\omega t]=\tfrac{p}{p^2+\omega^2}$$
$$Y=\tfrac{p}{(p^+2^2)(p^2+\omega^2)}$$
$$=\tfrac{1}{4-\omega^2}\left(\tfrac{p}{p^2+\omega^2}-\tfrac{p}{p^2+4}\right)$$

$$y(t) = \frac{1}{4-\omega^2}(\cos\omega t - \cos 2t)$$

3.8 用延拓法求解如下半无界问题

3.8.1

例：$u_t = a^2 u_{xx}, 0 < x < \infty, t > 0$

$u|_{t=0} = \varphi(x), 0 \le x < \infty$

$u_x|_{x=0} = f_1(t), 0 \le x < \infty.$

答:

考虑构建一个奇延拓, $u(-x,t) = -u(x,t)$,使其在全空间上定义

同时对 $\varphi(x)$ 进行奇延拓

$$\Phi(x) = \begin{cases} \varphi(x) \text{ if } x \ge 0 \\ -\varphi(-x) \text{ if } x < 0 \end{cases}$$

在 $-\infty < x < \infty$ 上解初值问题

$$u_t = a^2 u_{xx}, u|_{t=0} = \Phi(x)$$

对 $u(x,t), \Phi(x)$ 作关于 x 的 Fourier 变换

$$U(\lambda,t) = \mathcal{F}[u(x,t)], \hat{\Phi}(\lambda) = \mathcal{F}[\Phi(x)]$$

得到

$$U_t = -a^2\lambda^2 U$$
$$U = \hat{\Phi}(\lambda)e^{-a^2\lambda^2 t}$$
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\Phi}(\lambda)e^{-a^2\lambda^2 t}e^{i\lambda x}\,\mathrm{d}\lambda$$

3.8.2

例：$u_{tt} = a^2 u_{xx} + f(x,t), 0 < x < \infty, t > 0$

$u|_{t=0} = \varphi(x), 0 \le x < \infty$

$u_t|_{t=0} = \psi(x), 0 \le x < \infty$

$u_x|_{x=0} = 0, 0 \le x < \infty.$

答:

考虑构建一个偶延拓, $u(-x,t) = u(x,t)$,使其在全空间上定义 同时对 $\varphi(x), \psi(x)$ 进行偶延拓

$$\Phi(x) = \begin{cases} \varphi(x) \text{ if } x \ge 0 \\ \varphi(-x) \text{ if } x < 0 \end{cases}$$
$$\Psi(x) = \begin{cases} \psi(x) \text{ if } x \ge 0 \\ \psi(-x) \text{ if } x < 0 \end{cases}$$

于是在 $-\infty < x < +\infty$ 上,方程是一个有源波动方程，用达朗贝尔公式得

$$u(x,t) = \frac{1}{2}\varphi(x+at) + \frac{1}{2}\varphi(x-at) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t+\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

第四章 波动方程

4.1 通解法

4.1.1

例: $\begin{cases} 3u_{xx}+10u_{xy}+3u_{yy}=0 \text{ if} -\infty<x<+\infty \wedge y>0 \\ u|_{y=0}=\varphi(x) \text{ if } -\infty<x<+\infty \\ u_y|(y=0)=\psi(x) \text{ if } -\infty<x<+\infty \end{cases}$

答:

$$u_{xx}+\frac{10}{3}u_{xy}+u_{yy}=0$$

形如

$$u_{xx}-(A+B)u_{xy}+ABu_{yy}=0$$

则

$$A=-3, B=-\frac{1}{3}$$

取

$$\xi=y-3x, \eta=y-\frac{1}{3}x$$
$$u_x=-3\frac{\partial u}{\partial \xi}-\frac{1}{3}\frac{\partial u}{\partial \eta}$$
$$u_y=\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$
$$u_{xy}=-3\frac{\partial^2 u}{\partial \xi^2}-\frac{1}{3}\frac{\partial^2 u}{\partial \eta^2}-\frac{10}{3}\frac{\partial^2 u}{\partial \xi \partial \eta}$$
$$u_{xx}=9\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{9}\frac{\partial^2 u}{\partial \eta^2}+2\frac{\partial^2 u}{\partial \xi \partial \eta}$$
$$u_{yy}=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

代入化简得

$$\frac{\partial^2 u}{\partial \xi \partial \eta}=0$$
$$u(x,y)=F(\xi)+G(\eta)=F(y-3x)+G\left(y-\frac{1}{3}x\right)$$
$$u(x,0)=F(-3x)+G\left(-\frac{1}{3}x\right)=\varphi(x)$$
$$u_y(x,y)=F'(-3x)+G'\left(-\frac{1}{3}x\right)=\psi(x)$$

对两边从 0 到 x 积分

$$\int_0^x \psi(\xi)\,\mathrm{d}\xi=-\frac{1}{3}F(-3x)|_0^x-3G\left(-\frac{1}{3}x\right)|_0^x$$
$$u(x,y)=\frac{9}{8}\varphi\left(x-\frac{1}{3}y\right)-\frac{1}{8}\varphi(x-3y)+\frac{3}{8}\int_0^{x-\frac{1}{3}y}\psi(\xi)\,\mathrm{d}\xi-\frac{3}{8}\int_0^{x-3y}\psi(\xi)\,\mathrm{d}\xi$$

4.1.2

例: $u_{tt}=a^2u_{xx}$

$u|_{t=0}=0$

$u_t|_{t=0}=1$.

答:

达朗贝尔公式

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$
$$=\frac{1}{2a}\int_{x-at}^{x+at}\mathrm{d}\xi$$
$$=t$$

4.1.3

例: 在半平面 $\{(x,t)|-\infty<x<+\infty,t>0\}$,求弦振动方程 $u_{tt}=u_{xx}$, $M(2,5)$ 的依赖区间,它是否落在 $(1,0)$ 的影响区域内?

答:

作特征线

$$x+t=2+5=7, x-t=2-5=-3$$

$M(2,5)$ 的依赖区间

$$[-3,7]$$

作特征线

$$x+t=1, x-t=1$$

当 $t=5$ 时

$$x_1=-4, x_2=6$$

$[-3,6]$ 落在影响区域内, $(6,7]$ 不在影响区域内.

$u_{tt}=a^2\Delta u,(x,y)\in R^2,t>0$

$u|_{t=0}=3x+2y$

例: $u_t|_{t=0}=0$.

答:

取

$$\xi=x+r\cos\theta, \eta=y+r\sin\theta$$

在极坐标下,有

$$u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{\varphi(\xi,\eta)}{\sqrt{(at)^2-r^2}}r\,\mathrm{d}r$$

$$+\frac{1}{2\pi a}\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{\psi(\xi,\eta)}{\sqrt{(at)^2-r^2}}r\,\mathrm{d}r$$

$$u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{r(3x+2y)+3r^2\cos\theta+2r^2\sin\theta}{\sqrt{(at)^2-r^2}}\,\mathrm{d}\theta\,\mathrm{d}r$$

$$=\frac{3x+2y}{2a}\frac{\partial}{\partial t}\int_0^{at}\frac{\mathrm{d}r^2}{\sqrt{(at)^2-r^2}}$$

$$=3x+2y$$

4.1.4

例: 在 $t=0$ 平面上以 $(0,0)$ 为圆心, 1 为半径的圆域内,给出充分光滑的函数 φ,ψ,试问能否决定初值问题的解在 $(x,y,t)=\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ 的值?

$u_{tt}=a^2\Delta u,$

$u|_{t=0}=\varphi(x,y), u_t|_{t=0}=\psi(x,y)$

答:

点 $\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ 在 $t=\frac{1}{2}$ 时的传播半径为 $R=at=\frac{a}{2}$

该点距离 $(0,0)$ $r=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=1$

影响区域 $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{\sqrt{3}}{2}\right)^2\le\left(\frac{a}{2}\right)^2$

当 $a\ge 4$ 时, φ,ψ 一定落在区域内,可确定 $u\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ 的值,反之不能确认.

4.1.5

例：证明方程

$\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right]=\frac{1}{a^2}\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}$

的通解为 $u=\frac{1}{h-x}[F(x-at)+G(x+at)]$

答:

取

$$v(x,t)=(h-x)u(x,t)$$

则

$$\frac{\partial}{\partial x}v=-u+(h-x)\frac{\partial}{\partial x}u$$
$$\frac{\partial^2}{\partial x^2}v=-\frac{\partial}{\partial x}u-\frac{\partial}{\partial x}u+(h-x)\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial}{\partial x}\left[\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right]=-\frac{2}{h}\left(1-\frac{x}{h}\right)\frac{\partial u}{\partial x}+\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial x^2}=\frac{h-x}{h^2}\frac{\partial^2 v}{\partial x^2}$$
$$\frac{1}{a}\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}=\frac{1}{a^2}\frac{h-x}{h^2}\frac{\partial^2 v}{\partial t^2}$$

则原方程可以化为

$$\frac{\partial^2 v}{\partial t^2}=a^2\frac{\partial^2 v}{\partial x^2}$$

解得

$$v=F(x-at)+G(x+at)$$
$$u=\frac{1}{h-x}[F(x-at)+G(x+at)]$$

$F(x),G(x)$ 由初值条件 φ,ψ 确定.

4.2 求初值问题

4.2.1

例：$u_{tt}=a^2u_{xx}+x^2-a^2t^2,-\infty<x<+\infty,t>0$

$u|_{t=0}=0,u_t|_{t=0}=0$.

答:

由达朗贝尔公式有

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$
$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

取 $\frac{\partial}{\partial t}P(x,t,\tau)|_{t-\tau=0}=x^2-a^2t^2$ 有

$$\begin{cases}P_{tt}=a^2P_{xx}\\P|_{t-\tau=0}=0\\P_t|_{t-\tau=0}=x^2-a^2t^2\end{cases}$$
$$P(x,t,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}(\xi^2-a^2t^2)\,\mathrm{d}\xi$$
$$=x^2(t-\tau)+\frac{a^2(t-\tau)^3}{3}-a^2t^2(t-\tau)$$

再对 $P(x,t,\tau)$ 积分

$$u(x,t)=\int_0^t P(x,t,\tau)\,\mathrm{d}\tau$$
$$=\frac{1}{2}x^2t^2-\frac{5}{12}a^2t^4$$

第五章 椭圆型方程

5.1 导出散度公式和 Green 公式

取 $\boldsymbol{A}(x,y)=M\boldsymbol{i}+N\boldsymbol{j},\boldsymbol{n}=\cos\alpha\boldsymbol{i}+\cos\beta\boldsymbol{j}$

$$\iint_D \frac{\partial M}{\partial x}\,\mathrm{d}\sigma=\int_C M\boldsymbol{i}\cdot\boldsymbol{n}\,\mathrm{d}l$$

$$\iint_D \frac{\partial N}{\partial x}\,\mathrm{d}\sigma=\int_C N\boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}l$$

则

$$\iint_D\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial x}\right)\mathrm{d}\sigma=\int_C(M\boldsymbol{i}\cdot\boldsymbol{n}+N\boldsymbol{j}\cdot\boldsymbol{n})\,\mathrm{d}l$$

即

$$\iint_D \nabla\cdot\boldsymbol{A}\,\mathrm{d}\sigma=\int_C \boldsymbol{A}\cdot\boldsymbol{n}\,\mathrm{d}l$$

取

$$\boldsymbol{A}=\nabla\cdot u$$

$$\nabla(\nabla\cdot u)=\Delta u$$

$$\nabla u\cdot\boldsymbol{n}=\frac{\partial u}{\partial n}$$

代入化简得

$$\iint_D \Delta u\,\mathrm{d}\sigma=\int_C \frac{\partial u}{\partial n}\,\mathrm{d}l$$

在上式中取 $M=u\frac{\partial v}{\partial x},N=u\frac{\partial v}{\partial y}$

$$\iint_D\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial x}\right)\mathrm{d}\sigma$$

$$=\iint_D\left(uv_{xx}+uv_{yy}\right)\mathrm{d}\sigma+\iint_D\left(u_xv_x+u_yv_y\right)\mathrm{d}\sigma$$

$$=\iint_D(u\Delta v)\,\mathrm{d}\sigma+\iint_D(\nabla u\cdot\nabla v)\,\mathrm{d}\sigma$$

$$\int_C(M\boldsymbol{i}\cdot\boldsymbol{n}+N\boldsymbol{j}\cdot\boldsymbol{n})\,\mathrm{d}l$$

$$=\int_C\left(uv_x\boldsymbol{i}+uv_y\boldsymbol{j}\right)\cdot\boldsymbol{n}\,\mathrm{d}l$$

$$=\int_C u\frac{\partial v}{\partial n}\,\mathrm{d}l$$

即

$$\iint_D(u\Delta v)\,\mathrm{d}\sigma+\iint_D(\nabla u\cdot\nabla v)\,\mathrm{d}\sigma=\int_C u\frac{\partial v}{\partial n}\,\mathrm{d}l$$

交换 u,v

$$\iint_D(v\Delta u)\,\mathrm{d}\sigma+\iint_D(\nabla v\cdot\nabla u)\,\mathrm{d}\sigma=\int_C v\frac{\partial u}{\partial n}\,\mathrm{d}l$$

两式相减有

$$\iint_D(u\Delta v-v\Delta u)\,\mathrm{d}\sigma=\int_C\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)\mathrm{d}l$$

5.2

5.2.1

例: 设二维函数 $u(x,y)$ 在 ∂D 为边界的区域 D 内调和,且 $u\in C^2(\Omega)\cap C^0\left(\bar{\Omega}\right)$,证明

$$\int_C \frac{\partial u}{\partial n}\,\mathrm{d}l=0$$

答:

在已导出的散度定理中,

$$\iint_D \Delta u\,\mathrm{d}\sigma=\int_C \frac{\partial u}{\partial n}\,\mathrm{d}l$$

有

$$\Delta u = 0$$

即

$$\int_C \frac{\partial u}{\partial n}\,\mathrm{d}l = 0$$

5.2.2

例：三维 Poisson 方程的 Dirichlet 问题,适用 Green 函数表示该问题的解

$\Delta u = -f(M), M \in \Omega$

$u|_{\partial\Omega} = 0$.

答:

在 Ω 区域中

$$\Delta_M G(M, M') = -\delta(M - M'), M, M' \in \Omega$$

其中 $\delta(M - M')$ 是三维 Dirac delta 函数

在 $\partial\Omega$ 边界上

$$G(M, M') = 0, M \in \partial\Omega$$

则

$$u(M) = \int_\Omega G(M, M') f(M')\,\mathrm{d}V(M')$$

5.2.3

例：在平面 $-\infty < x < +\infty, y > 0$ 上求 Green 函数 $G(M, M_0)$

$-\Delta G = \delta(x - x_0, y - y_0),$

$G|_{\partial\Omega} = 0$

答:

在整个平面上 Laplace 方程的 Green 函数为：

$$G_0(M, M_0) = -\frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

由于边界条件要求 $G(x, 0) = 0$,可以通过镜像法构造一个辅助源点 $M_0' = (x_0, -y_0)$ 并定义

$$G(M, M_0) = G_0(M, M_0) - G_0(M, M_0')$$

$$G_0(M, M_0') = -\frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (y + y_0)^2}$$

易得 $G(M, M_0)$ 在区域上调和,且在边界上等于 0

$$G(M, M_0) = -\frac{1}{2\pi}\ln\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{\sqrt{(x-x_0)^2+(y+y_0)^2}}$$

5.2.4

例：在平面 $-\infty < x < +\infty, y > 0$ 上求解

$\Delta u = -f(M), -\infty < x < +\infty, y > 0$

$u|_{y=0} = \varphi(x)$

答:

做 Laplace 变换

$$U(\lambda, y) = \mathcal{F}[u(x, y)]$$

$$F(\lambda, y) = \mathcal{F}[f(x, y)]$$

$$\Phi(\lambda) = \mathcal{F}[\varphi(x)]$$

则

$$U_{yy} - \lambda^2 U = -F$$

$$U(\lambda, y) = \Phi(\lambda)e^{-|\lambda| y} + \int_0^y G(\lambda, y - \xi)F(\lambda, \xi)\,\mathrm{d}\xi$$

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\Phi(\lambda)e^{-|\lambda|y}+\int_0^y G(\lambda,y-\xi)F(\lambda,\xi)\,\mathrm{d}\xi\right]e^{i\lambda x}\,\mathrm{d}\lambda$$

5.2.5

例: 在圆域 $x^2+y^2<R^2$ 上求解

$$\Delta u=-f(M), r<R$$

$$u|_{r=R}=\varphi$$

答:

由达朗贝尔公式有

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi$$

$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

取 $\frac{\partial}{\partial t}P(x,t,\tau)|_{t-\tau=0}=x^2-a^2t^2$ 有

$$\begin{cases}P_{tt}=a^2P_{xx}\\P|_{t-\tau=0}=0\\P_t|_{t-\tau=0}=x^2-a^2t^2\end{cases}$$

$$P(x,t,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}(\xi^2-a^2t^2)\,\mathrm{d}\xi$$

$$=x^2(t-\tau)+\frac{a^2(t-\tau)^3}{3}-a^2t^2(t-\tau)$$

再对 $P(x,t,\tau)$ 积分

$$u(x,t)=\int_0^t P(x,t,\tau)\,\mathrm{d}\tau$$

$$=\frac{1}{2}x^2t^2-\frac{5}{12}a^2t^4$$