StudyThinking

卷二 数学物理方程笔记

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第一章 Fourier Transform

1.1 definition

$$F(\lambda)=\mathcal{F}[f(x)]=\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,\mathrm{d}x$$
$$f(x)=\mathcal{F}^{-1}[f(x)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda$$

- $\mathcal{F}\left[e^{-a|x|}\right], a>0$

$$\begin{aligned}\mathcal{F}\left[e^{-a|x|}\right]&=\int_{-\infty}^{\infty} e^{-a|x|}e^{-i\lambda x}\,\mathrm{d}x\\&=\int_{-\infty}^{0} e^{ax-i\lambda x}\,\mathrm{d}x+\int_{0}^{\infty} e^{-ax-i\lambda x}\,\mathrm{d}x\\&=\frac{1}{a-i\lambda}e^{(a-i\lambda)x}|_{-\infty}^{0}-\frac{1}{a+i\lambda}e^{-(a+i\lambda)x}|_{0}^{\infty}\\&=\frac{1}{a-i\lambda}+\frac{1}{a+i\lambda}\\&=\frac{2a}{a^2+\lambda^2}\end{aligned}$$

$* \lim_{t\to\infty} e^{-(\beta+i\lambda)t}=0, \beta>0$

- $\mathcal{F}\left[e^{-ax^2}\right], a>0$

$$\begin{aligned}\mathcal{F}\left[e^{-ax^2}\right]&=\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\lambda x}\,\mathrm{d}x\\&=\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2-\frac{\lambda^2}{4a}}\,\mathrm{d}x\\&=e^{-\frac{\lambda^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2}\,\mathrm{d}x\\&=\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}\end{aligned}$$

$*$Gaussian integral: $\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}$

- $\mathcal{F}[\cos\eta x^2], \eta>0$

$$\begin{aligned}\mathcal{F}[\cos\eta x^2]&=\int_{-\infty}^{\infty}\cos\eta x^2 e^{-i\lambda x}\,\mathrm{d}x\\&=\int_{-\infty}^{\infty}\frac{e^{-i\eta x^2}+e^{i\eta x^2}}{2}e^{-i\lambda x}\,\mathrm{d}x\\&=\frac{1}{2}\left(\int_{-\infty}^{\infty} e^{i\eta\left(x-\frac{\lambda}{2\eta}\right)^2-i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x+\int_{-\infty}^{\infty} e^{-i\eta\left(x+\frac{\lambda}{2\eta}\right)^2+i\frac{\lambda^2}{4\eta}}\,\mathrm{d}x\right)\\&=\frac{1}{2}e^{-i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{i\frac{\pi}{4}}+\frac{1}{2}e^{i\frac{\lambda^2}{4\eta}}\sqrt{\frac{\pi}{\eta}}e^{-i\frac{\pi}{4}}\\&=\frac{1}{2}\sqrt{\frac{\pi}{\eta}}\left[\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)-i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)+i\sin\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)\right]\\&=\sqrt{\frac{\pi}{\eta}}\cos\left(\frac{\lambda^2}{4\eta}-\frac{\pi}{4}\right)\end{aligned}$$

- $\mathcal{F}[\sin\eta x^2], \eta>0$

$$\mathcal{F}[\sin\eta x^2]=\sqrt{\frac{\pi}{\eta}}\sin\left(\frac{\lambda^2}{4\eta}+\frac{\pi}{4}\right)$$

$*$Euler's formula : $e^{ix}=\cos x+i\sin x$

$*$Gaussian-like integrals: $\int_{-\infty}^{\infty} e^{\pm i\eta x^2}\,\mathrm{d}x=\sqrt{\frac{\pi}{\eta}}e^{\pm i\frac{\pi}{4}}, \eta>0$

1.2 properties

1.2.1 linear property

$$\mathcal{F}[\alpha f(x)+\beta g(x)]=\alpha\mathcal{F}[f(x)]+\beta\mathcal{F}[g(x)]$$

1.2.2 shift property

$$\mathcal{F}[f(x-b)] = e^{-i\lambda b}\mathcal{F}[f(x)]$$

1.2.3 differentiation property

$$\mathcal{F}[f'(x)] = i\lambda\mathcal{F}[f(x)]$$
$$\mathcal{F}\left[f^{(n)}(x)\right] = (i\lambda)^n\mathcal{F}[f(x)]$$
$$\mathcal{F}[xf(x)] = i\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[f(x)]$$
$$\mathcal{F}[x^n f(x)] = i^n\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}[f(x)]$$

1.2.4 convolution property

$$\mathcal{F}[f(x)*g(x)] = \mathcal{F}[f(x)]\cdot\mathcal{F}[g(x)]$$
$$\mathcal{F}[f(x)\cdot g(x)] = \tfrac{1}{2\pi}\mathcal{F}[f(x)]*\mathcal{F}[g(x)]$$

1.3 function δ

1.3.1 definition

$$\delta(x-x_0) = \begin{cases}\infty \text{ if } x=x_0\\ 0 \text{ else}\end{cases}$$
$$\int_{-\infty}^{\infty}\delta(x-x_0) = 1$$

1.3.2 properties

$$\delta(x) = \delta(-x)$$
$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$$
$$\delta(x)*f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,\mathrm{d}\xi = f(x)$$
$$\delta(x-a)*f(x) = \int_{-\infty}^{\infty} f(x-\xi)\delta(\xi-a)\,\mathrm{d}\xi = f(x-a)$$

1.3.3 fourier transform

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty}\delta(x-x_0)e^{-i\lambda x}\,\mathrm{d}x = e^{-i\lambda x}$$

when $x_0 = 0$

$$\mathcal{F}[\delta(x)] = 1$$
$$\delta(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\,\mathrm{d}\lambda$$

since $\delta(x) = \delta(-x)$

$$\delta(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}\lambda$$

the fourier transform of 1 can be given

$$\mathcal{F}[1] = 2\pi\delta(\lambda)$$

第二章 Laplace Transform

2.1 definition

$$F(p) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$$

- $\mathcal{L}[1]$

$$\mathcal{L}[1] = \int_0^\infty 1 \cdot e^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}e^{-pt}|_0^\infty$$
$$= \tfrac{1}{p}$$

- $\mathcal{L}[t]$

$$\mathcal{L}[t] = \int_0^\infty te^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}te^{-pt}|_0^\infty + \tfrac{1}{p}\int_0^\infty e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{p^2}$$

- $\mathcal{L}[e^{at}]$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-pt}\,\mathrm{d}t$$
$$= \int_0^\infty e^{-(p-a)t}\,\mathrm{d}t$$
$$= \tfrac{1}{p-a}$$

- $\mathcal{L}[\cos\omega t]$

$$\mathcal{L}[\cos\omega t] = \int_0^\infty \cos\omega t e^{-pt}\,\mathrm{d}t$$
$$= -\tfrac{1}{p}\cos\omega t e^{-pt}|_0^\infty - \tfrac{\omega}{p}\int_0^\infty \sin\omega t e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{p} + \tfrac{\omega}{p^2}\sin\omega t e^{-pt}|_0^\infty - \tfrac{\omega^2}{p^2}\mathcal{L}[\cos\omega t]$$
$$= \tfrac{p}{p^2+\omega^2}$$

- $\mathcal{L}[\sin\omega t]$

$$\mathcal{L}[\sin\omega t] = \int_0^\infty \sin\omega t e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{2i}\int_0^\infty (e^{i\omega t} - e^{-i\omega t})e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{2i}\int_0^\infty \left(e^{-(p-i\omega)t} - e^{-(p+i\omega)t}\right)\mathrm{d}t$$
$$= \tfrac{1}{2i}\left(\tfrac{1}{p-i\omega} - \tfrac{1}{p+i\omega}\right)$$
$$= \tfrac{\omega}{p^2+\omega^2}$$

- $\mathcal{L}[\cosh\omega t]$

$$\mathcal{L}[\cosh\omega t] = \int_0^\infty \cosh\omega t e^{-pt}\,\mathrm{d}t$$
$$= \tfrac{1}{2}\int_0^\infty \left(e^{-(p-\omega)t} + e^{-(p+\omega)t}\right)\mathrm{d}t$$
$$= \tfrac{1}{2}\left(\tfrac{1}{p-\omega} + \tfrac{1}{p+\omega}\right)$$
$$= \tfrac{p}{p^2-\omega^2}$$

- $\mathcal{L}[\sinh\omega t]$

$$\mathcal{L}[\sinh \omega t] = \frac{\omega}{p^2 - \omega^2}$$

2.2 properties

2.2.1 linear property

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

2.2.2 shift property

$$\mathcal{L}[e^{at} f(t)] = F(p - a)$$

2.2.3 differentiation property

$$\mathcal{L}[f'(t)] = p\mathcal{L}[f(t)] - f(0)$$
$$\mathcal{L}\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1} f(0) - \ldots - p f^{(n-2)}(0) - f^{(n-1)}(0)$$
$$\mathcal{L}[t f(t)] = -\frac{\mathrm{d}}{\mathrm{d}p} F(p)$$
$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}p^n} F(p)$$

2.2.4 convolve property

$$f(t) * g(t) = \int_0^t f(t) g(t - \tau) \, \mathrm{d}\tau$$
$$\mathcal{L}[f(t) * g(t)] = F(p) \cdot G(p)$$

第三章 Fundamental Equations

3.1 wave equation

$$u_{tt} = a^2 u_{xx}$$
$$u_{tt} = a^2\left(u_{xx} + u_{yy}\right)$$
$$u_{tt} = a^2\left(u_{xx} + +u_{yy} + u_{zz}\right)$$

3.2 wave equation with a source term

$$u_{tt} = a^2 u_{xx} + f(x,t)$$
$$u_{tt} = a^2\left(u_{xx} + u_{yy}\right) + f(x,y,t)$$
$$u_{tt} = a^2\left(u_{xx} + +u_{yy} + u_{zz}\right) + f(x,y,z,t)$$

3.3 heat equation

$$u_t = a^2 u_{xx}$$
$$u_t = a^2\left(u_{xx} + u_{yy}\right)$$
$$u_t = a^2\left(u_{xx} + +u_{yy} + u_{zz}\right)$$

3.4 heat equation with a source term

$$u_t = a^2 u_{xx} + f(x,t)$$
$$u_t = a^2\left(u_{xx} + u_{yy}\right) + f(x,y,t)$$
$$u_t = a^2\left(u_{xx} + +u_{yy} + u_{zz}\right) + f(x,y,z,t)$$

3.5 classification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$*$A, B, C, D, E, F and G are functions of x and y, but not of u.

$$\Delta = B^2 - AC$$

$\Delta > 0$(hyperbolic equation)

$$u_{xy} = [\cdots], u_{xx} - u_{yy} = [\cdots]$$

$\Delta = 0$(parabolic equation)

$$u_{xx} = [\cdots], u_{yy} = [\cdots]$$

$\Delta < 0$(elliptic equation)

$$u_{xx} + u_{yy} = [\cdots]$$

$*$ $[\cdots]$ represents all terms that do not contain second-order partial derivatives.

3.6 simplification of second-order partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

characteristic equation:

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$$

characteristic lines are determined by the solutions of the characteristic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$
$$\begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$
$$\xi(x,y) = c_1, \eta(x,y) = c_2$$

$(1)\Delta = B^2 - AC > 0$

e.g. $u_{xx} - 4u_{xy} + u_{yy} = 0$

$$A = 1, B = -2, C = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2 \pm \sqrt{3}$$
$$y + \left(2 \pm \sqrt{3}\right)x = c$$
$$\xi(x,y) = y + \left(2 + \sqrt{3}\right)x, \eta(x,y) = y + \left(2 - \sqrt{3}\right)x$$

$(2)\Delta = 0$

Characteristic lines $\xi(x,y) = c_1$ are still governed by the characteristic equation, while $\eta(x,y)$ can be any function independent of $\xi(x,y)$, provided that the Jacobian determinant is not equal to zero.

$(3)\Delta < 0$

e.g. $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + u_y = 0$

$$A = 1, B = 2, C = 5$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \pm i$$
$$2x - y \pm ix = c$$
$$\xi(x,y) = x, \eta(x,y) = 2x - y$$

3.7 constant-coefficient equation

when $A, B, C \in \mathbb{R}$

$(1)\Delta > 0$

$$\xi = y - \frac{B + \sqrt{B^2 - AC}}{A}x, \eta = y - \frac{B - \sqrt{B^2 - AC}}{A}x$$

$(2)\Delta = 0$

$$\xi = y - \frac{B}{A}x, \eta = y$$

$(3)\Delta < 0$

$$\xi = y - \frac{B}{A}x, \eta = \frac{\sqrt{AC - B^2}}{A}x$$

e.g. $u_{xx} - (A+B)u_{xy} + ABu_{yy} = 0$

$$\xi = y + Ax, \eta = y + Bx$$
$$u_{xx} - (A+B)u_{xy} + ABu_{yy} = -(A-B)^2 u_{\xi\eta}$$

第四章 Separation of Variables

4.1 string vibration equation

$$u_{tt} = a^2 u_{xx}(0 < x < l, t > 0)$$
$$u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x)(0 \leq x \leq l)$$
$$u|_{x=0} = 0, u|_{x=l} = 0(t > 0)$$

assume the equation has a solution in the form of separated variables：

$$u(x,t) = X(x)T(t)$$

substitute into the equation:

$$X(x)T''(t) = a^2 X''(x)T(t)$$
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2T(t)} = -\lambda$$

obtain ordinary differential equations for the spatial function and the temporal function

$$X''(x) + \lambda X(x) = 0$$
$$T''(t) + \lambda a^2 T(t) = 0$$

from the boundary conditions $X(0) = X(l) = 0$:

(1)$\lambda \leq 0$

only the trivial solution

(2)$\lambda > 0$

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

since $X(0) = X(l) = 0$

$$c_1 = 0$$
$$c_2 = 1, \sqrt{\lambda} = \frac{k\pi}{l}, k = 1,2,3,...$$
$$X(x) = \sin\frac{k\pi}{l}x$$

similarly:

$$T_k(t) = A_k \cos\frac{k\pi a}{l}t + B_k \sin\frac{k\pi a}{l}t$$

by the principle of superposition:

$$u(x,t) = \sum_{k=1}^{\infty}\left(A_k \cos\frac{k\pi a}{l}t + B_k \sin\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x$$
$$u_t(x,t) = \sum_{k=1}^{\infty}\left(-A_k\frac{k\pi a}{l}\sin\frac{k\pi a}{l}t + B_k\frac{k\pi a}{l}\cos\frac{k\pi a}{l}t\right)\sin\frac{k\pi}{l}x$$

since $u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x)$

$$\varphi(x) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x$$
$$\psi(x) = \sum_{k=1}^{\infty} A_k\frac{k\pi a}{l}\sin\frac{k\pi}{l}x$$
$$A_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$
$$A_k = \frac{2}{k\pi a}\int_0^l \psi(x)\sin\frac{k\pi}{l}x\,\mathrm{d}x$$

conditions for the application of the method of separation of variables:

(1) The general equation must be linear.

(2) The general equation must be homogeneous.

(3) The boundary conditions must be homogeneous.

4.2 heat conduction equation

$$u_t = a^2 u_{xx}(0 < x < l, t > 0)$$
$$u|_{t=0} = \varphi(x)(0 \leq x \leq l)$$

$$u_x|_{x=0} = 0, u_x|_{x=l} = 0(t > 0)$$

apply separation of variables:

$$X''(x) + \lambda X(x) = 0, X'(0) = X'(l) = 0$$
$$T'(t) + \lambda a^2 T(t) = 0$$

$(1)\lambda < 0$

only the trival solution

$(2)\lambda = 0$

$$X_0 = 1, \lambda_0 = 0$$

$(2)\lambda > 0$

$$X_k = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$
$$X_k' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

since $X'(0) = X'(l) = 0$

$$c_1 = 1, c_2 = 0$$
$$X_k = \cos\tfrac{k\pi}{l}x, \lambda_k = \left(\tfrac{k\pi}{l}\right)^2, k = 1, 2, ...$$

similarly:

$$T_0 = A_0, T_k = A_k e^{-\left(\frac{k\pi a}{l}\right)^2 x}$$

by the principle of superposition:

$$u(x,t) = A_0 + \textstyle\sum_{k=1}^{\infty} A_k e^{-\left(\frac{k\pi a}{l}\right)^2 x} \cos\tfrac{k\pi}{l}x$$

since $u|_{t=0} = \varphi(x)$

$$A_0 = \tfrac{1}{l}\textstyle\int_0^l \varphi(x)\,\mathrm{d}x$$
$$A_k = \tfrac{2}{l}\textstyle\int_0^l \varphi(x)\cos\tfrac{k\pi}{l}x\,\mathrm{d}x$$